INTRODUCTION

When explaining Faraday's law to a class of undergraduate electrical-engineering students, one is naturally drawn to treating the ideal transformer. Referring to Figure 1, we could proceed as follows: a toroidal core is assumed, and for simplicity we will use one with a rectangular cross section. A primary consisting of $N$ finely spaced turns carries a sinusoidal current with phasor value $I_0$, resulting in a total phasor magnetic flux, $\Phi$, threading in the azimuthal direction around the core; It is invariably assumed that the magnetic-flux-density vector, $B$, is zero except within the core. For simplicity, we assume the transformer to have a secondary with a single turn of perfectly conducting wire forming a rectangular loop. There is a gap in this loop, across which we seek the phasor voltage, $V$. The equation of Maxwell known in integral form as Faraday's induction law is applied to this problem, with the result

$$V = \oint \mathbf{E} \cdot d\mathbf{l} = \iint -j\omega \mathbf{B} \cdot d\mathbf{S} = -j\omega \Phi. \quad (1)$$

Often one would be using a multi-turn secondary, and at this point one would discuss turns ratio, and relate the driving input voltage, $V_0$, to the output voltage, $V$. 

![Diagram of a toroidal core with a rectangular cross section and a single turn of perfectly conducting wire forming a rectangular loop.](image)
If the class is bright and the teacher is lucky, a student will now raise his or her hand and say something like, Professor, you've contradicted yourself. You assumed that there is no magnetic-flux density, $\mathbf{B}$, outside the core. Using Maxwell's equation
$$\nabla \times \mathbf{B} = j\omega\mu_0\varepsilon_0\mathbf{E}$$
outside the core, you can see that the electric field must vanish, yet you have essentially integrated this zero electric field around a loop and gotten a nonzero result. How can this be?

It is remarkable how few undergraduate-level textbooks deal with this question. The only one that the author has found that provides a clue to the necessary answer is Hammond [1], who treated a problem similar to that in Figure 1 and concluded, “There must be some magnetic field outside the toroid.” A book on the philosophy of physics, Lange [2], presents - without resolving - a similar apparent contradiction. The purpose of the present paper is to provide a teaching tool for the instructor in electromagnetism who might encounter a question like the one just given, or who might have wondered about the answer, simply through his or her own speculation.

We study here a transformer carrying an alternating current at power-line frequencies, and provide calculations of the electric field both inside and outside the toroidal core, as well as the corresponding magnetic fields. We also make a computation of $\oint \mathbf{E} \cdot \mathbf{dl}$ using the numerically derived electric field, $\mathbf{E}$, outside the toroidal core. We show that for these frequencies and for a transformer with dimensions that are of the order of a meter, the
result is extremely well approximated by the use of the magnetic-flux expression contained on the right in Equation (1). This equation, which will be shown to be an approximation for the secondary voltage, is justified by our comparing the orders of magnitude of the magnetic fields inside and outside the core of the transformer. Thus, we now have all the material needed to deal with the hypothetical student's question. Finally, suggestions are made as to how one might incorporate all of the preceding calculations into a homework assignment that a student could carry out, using MATLAB or a comparable computational package.

**Calculations**

Shown in Figure 2 is a cross section of the toroidal core. We call the inner radius \( a \), and take the outer radius to be \( b \). Thus, the cross section has width \( b - a \). The overall length of the core is \( L \). The \( N \) tightly wound turns of wire on the core, each carrying a constant phasor current \( I_0 \), effectively create: a surface current \( I = N I_0 \) amperes, circulating on the surface of the toroid in the direction shown by the arrow. For simplicity, the core is assumed to be nonmagnetic. We make no assumptions about the conductivity of the wire. However, if the total length of conductor around the toroid is very long, one should be concerned about non-uniformity in the current along the wire's length. To achieve uniformity, One could conceivably place a separate identical generator in each turn. As a further guarantee of uniformity in current, we must assume that \( 2\pi b/\lambda \ll 1 \) and that \( 2\pi L/\lambda << 1 \), where \( \lambda \) is the wavelength in free space of electromagnetic waves at the radian frequency, \( \omega \), of the generator.
An observer is at the point with cylindrical coordinates of \((r, \phi, z)\), where \(\phi\), an azimuthal angle, is measured in the same sense as the flux, \(\Phi\), shown in Figure 1. When \(\phi = 0\), we are on a line normal to’ and directed outward from the plane of the paper in Figure 2. A general, current-carrying, source point is located at \((r', \phi', z')\). The observer and source points are located by vectors \(\mathbf{r}\) and \(\mathbf{r}'\), which extend from the origin of the coordinates. When the current is confined to surfaces, the vector potential created is given by

\[
A(\mathbf{r}) = \frac{\mu_0}{4\pi} \iint K(\mathbf{r}') \frac{e^{-jkR}}{R} dS'
\]  

(2)

Here, \(K(\mathbf{r}')\) is the vector surface-current density at the source point, while \(R = |\mathbf{r} - \mathbf{r}'|\) is the physical distance between the source and observation points. The integration is taken over all current-carrying surfaces; their differential area is denoted by \(dS'\). It is a standard undergraduate exercise to show that the Pythagorean separation between the source point and the observer is
Since all fields and potentials in this problem are independent of the coordinate \( \phi \) we will set this parameter to zero in our calculations, and we will often denote observation points as having coordinates \((r, z)\).

The upward surface-current density is \( K_z = I/(2\pi a) \) on the inner cylindrical surface of the toroid (at radius \( a \)), and is \( K_z = -I/(2\pi b) \) on the outer cylindrical surface (at radius \( b \)). On these surfaces, \( dS' = a\,d\phi'\,dz' \) and \( dS' = bd\phi'\,dz' \), respectively. Across the top and bottom of the toroid (at \( z = \pm L/2 \)), there are radial surface currents, \( K_r = \pm NI/(2\pi r') \), where the plus and minus signs apply to the top and bottom, respectively. We have \( dS' = r'd\phi'\,dr' \) for both top and bottom. An application of the continuity equation to the surface currents shows that the electrical charge on the toroid is zero, which implies that the scalar electrical potential, \( V(r, \phi, z) \), created in space is also zero.

To apply Equation (2) we need the distance, \( R \), from the observation point \((r, z)\) or \((r, 0, z)\) to a source point on the surface of the toroid. With the source point on the inside surface of the toroid where \( r' = a \), from Equation (3) we have for this distance

\[
R = R_i(r, z, a) = \sqrt{r^2 + a^2 - 2ar\cos(\phi') + (z' - z)^2}.
\]

For a source point on the outside surface, \( r' = b \), we call this distance

\[
R = R_o = \sqrt{r^2 + b^2 - 2br\cos(\phi') + (z' - z)^2}.
\]
When the source point resides on the top of the toroid (see Figure 2), where \( z' = L/2 \), the required distance is

\[
R_t(r, z, L) = \sqrt{r^2 + (z - L/2)^2 - 2rr'\cos(\phi') + (z - L/2)^2}.
\]

The distance for a source point on the bottom of the toroid, where \( z' = -L/2 \), is

\[
R_b(r, z, L) = \sqrt{r^2 + (z + L/2)^2 - 2rr'\cos(\phi') + (z + L/2)^2}.
\]

The four subscripts \( i, o, t, \) and \( b \) suggest the words inner, outer, top, and bottom. We resolve Equation (2) into two scalar components at the observation point, and with these four preceding distances, we can state formulas for the two components of the magnetic vector potential:

\[
A_z(r, z) = \frac{\mu_0}{4\pi^2} \int_{0}^{\pi L/2} \int_{-L/2}^{L/2} \left( \frac{e^{-jkR_i}}{R_i} - \frac{e^{-jkR_o}}{R_o} \right) dz'd\phi'. \tag{4}
\]

and

\[
A_r(r, z) = \frac{\mu_0}{4\pi^2} \int_{0}^{\pi} \int_{0}^{b} \left( \frac{e^{-jkR_t}}{R_t} - \frac{e^{-jkR_b}}{R_b} \right) dr' \cos \phi' d\phi'. \tag{5}
\]

In deriving Equation (5), it is helpful to notice that at the observation point, where \( \phi' = 0 \), the vector component \( A_r(r, z) \) becomes identical to the Cartesian vector component \( A_x(r, z) \), where the positive \( x \) direction is perpendicular and upward from the plane of the page in Figure 2. Observe that we have taken advantage of a symmetry, \( \cos \phi' = \cos(-\phi') \), so that the above integrals over \( \phi' \) take place from 0 to \( \pi \). In the preceding equations, we use \( \mu_0 \) and \( \varepsilon_0 \) as the permeability and permittivity.
of free space, respectively, while \( k = \omega \sqrt{\mu_0 \varepsilon_0} \) is the usual free-space wavenumber.

Since the scalar electric potential, \( V \) is identically zero, the equation for the vector electric field \( \mathbf{E} = -\nabla V - j\omega \mathbf{A} \) yields the components

\[
E_z = -j\omega A_z ,
\]

\[
E_r = -j\omega A_r .
\]

Let us assume that we are using the usual North American powerline radian frequency of 60 Hz, so that \( \omega = 2\pi f = 120\pi \), and that the distances \( a, b, \) and \( L \) are of the order of a few meters or less. Then it is easily verified that the exponents in Equations (4) and (5) are no bigger than of the order of \( 10^{-5} \), and we feel confident in using finite Maclaurin-series approximations for the exponential functions. The student of undergraduate electromagnetic theory will recall the infinite series

\[
e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \frac{w^4}{4!} + \frac{w^5}{5!} ... ,
\]

which is valid for all complex \( w \). For our purposes, we will need at most the first four terms in the above.

Thus, we have with, \( w = -jkR \),

\[
e^{-jkR} \approx 1 - jkR - k^2 R^2/2 + jk^3 R^3/6 .
\]

The quantity \( R \) will be allowed to be each of the functions \( R_t, R_b, R_i, R_o \), which appear in the integrals in Equations (4) and (5). Using only the first term in the above series to approximate the exponentials in Equation (4) - which means that
we obtain $A_z(0, r, z)$, our first-order approximation to $A_z$:

\[
A_z(0, r, z) = \frac{\mu_0}{4\pi^2} \int_0^\infty \left[ \int_{-\infty}^\infty \left( \frac{1}{R_i} - \frac{1}{R_o} \right) dz' \right] d\phi'
\]

\[
= \frac{\mu_0}{4\pi^2} \int_0^\infty \left[ \int_{-\infty}^\infty \frac{1}{\sqrt{r^2 + a^2 - 2ar \cos(\phi') + (z' - z)^2}} \right] d\phi' \]

\[
- \frac{1}{\sqrt{r^2 + b^2 - 2br \cos(\phi') + (z' - z)^2}} \right] dz' \right] d\phi'.
\]

In a standard table of integrals we find [4, entry 200.01] that

\[
\int \frac{1}{\sqrt{\chi^2 + \alpha^2}} dx = \log \left( \chi + \sqrt{\chi^2 + \alpha^2} \right).
\]

This permits evaluation of the integrals over $z'$ in Equation (10).

If we take $\chi = (z' - z)$ and $\alpha^2 = r^2 + r'^2 - 2rr' \cos \phi'$, where $r'$ equals $a$ or $b$, as needed, we obtain
\[ A_{z0}(r, z) = \frac{\mu_0}{4\pi^2} I \int_0^\pi \log \left[ \frac{-L/2 - z + \sqrt{(-L/2 - z)^2 + r^2 + a^2 - 2ar \cos \phi'}}{-L/2 - z + \sqrt{(-L/2 - z)^2 + r^2 + a^2 - 2ar \cos \phi'}} \right] \, d\phi'. \]

All logs here are base \( e \). The use of Equation (11) in Equation (6) would result in a value of \( E_z \) that is purely imaginary. To obtain an initial approximation for the real part of \( E_z \), we should include the second term from the series in Equation (9) when we approximate exponentials in the integral in Equation (4). This is an imaginary expression that would yield an approximation to the real part of \( E_z \). Unfortunately, this second term contributes nothing to the value of the integral in Equation (4), as is easily verified by studying

\[ A_{z1}(r, z) = \frac{\mu_0}{4\pi^2} I \int_0^\pi \int_{-L/2}^{L/2} \left( \frac{-jkR_i}{R_i} \right) d\phi' \]

If we wish to obtain a nontrivial approximation to the real part of \( E_z \), we must use the very last term in Equation (9), i.e., \( jk^3 R^3 / 6 \), when we approximate the integrand in Equation (4). We obtain

\[ \text{Im}[A_z(r, z)] = \frac{\mu_0 k^3}{24\pi^2} I \int_0^\pi \int_{-L/2}^{L/2} \left( R_i^2 - R_o^2 \right) d\phi' \]

\[ = \frac{\mu_0 k^3}{24\pi^2} I \int_0^\pi \int_{-L/2}^{L/2} \left( a^2 - 2ar \cos \phi' - b^2 + 2br \cos \phi' \right) d\phi' \]

\[ = -\frac{\mu_0 k^3 (b^2 - a^2) L}{24\pi}. \]
The preceding result is independent of the coordinates of the observer. Summarizing from Equations (6), (11), and (12), we have

\[
E_z(r, z) = \frac{-k^4}{24\pi} \left( b^2 - a^2 \right) L \eta I + \frac{-jk\eta}{4\pi^2} \int_0^\pi \left[ P(a, \phi') - P(b, \phi') \right] d\phi'
\]

V/m,

(13a)

where

\[
P(a, \phi') = \log \left[ \frac{L/2 - z + \sqrt{(L/2 - z)^2 + r^2 + a^2 - 2ar \cos \phi'}}{-L/2 - z + \sqrt{(-L/2 - z)^2 + r^2 + a^2 - 2ar \cos \phi'}} \right]
\]

(13b)

and

\[
P(b, \phi') = \log \left[ \frac{L/2 - z + \sqrt{(L/2 - z)^2 + r^2 + b^2 - 2br \cos \phi'}}{-L/2 - z + \sqrt{(-L/2 - z)^2 + r^2 + b^2 - 2br \cos \phi'}} \right]
\]

(13c)

Here, \( \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \) is the intrinsic impedance of free space.

Why have we not elected to use the third term appearing on the right in the series of Equation (9), especially since we have employed all the other terms? For the physical lengths used here (on the scale of meters), and for power-line frequencies, inclusion of this term would result in a modification to the real part of \( A_z(r, z) \) that is about 10 orders of magnitude less than our initial approximation to the real part found from Equation (11). The effect of including this term would be masked by round-off errors arising when we numerically compute the integral in Equation (11), and so there is no point in our keeping
the term in the calculations. However, we will later see that this third term, small as it is, must be included in our work when we are seeking the magnetic (in contrast to the electric) field outside the core of the transformer.

The formulation of \( A_r(r, z) \) follows a similar route, but this time involves our using the series approximations for \( e^{-jkR_t} \) and \( e^{-jkR_b} \) in Equation (5), derived from Equation (9). Beginning with just the first term, for each exponential gives us

\[
\text{Re}(A_{r0}) = \frac{\mu_0 I}{4\pi^2} \int_0^b \left[ \int_a^b \left( \frac{1}{R_t} - \frac{1}{R_b} \right) dr' \right] \cos \phi' d\phi' = \frac{\mu_0 I}{4\pi^2} \int_0^b \left[ \int_a^b \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z - L/2)^2}} \right. \\
- \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z + L/2)^2}} \left. dr' \right] \cos \phi' d\phi'.
\]

The integration over \( r' \) can be accomplished with tables. From [4], entry 380.0011, we have

\[
\int \frac{d\chi}{(\alpha \chi^2 + \beta \chi + \gamma)^{1/2}} = \frac{1}{\alpha^{1/2}} \log \left| 2 \left[ \alpha \left( \alpha \chi^2 + \beta \chi + \gamma \right) \right]^{1/2} + 2\alpha \chi + \beta \right| \\
\text{for } \alpha > 0.
\]
Taking \( \chi = r', \alpha = 1, \beta = -2r \cos \phi', \gamma = r^2 + (z \pm L/2)^2 \), the reader should verify that

\[
\text{Re}(A_{r0}) = \frac{\mu_0 I}{4\pi^2} \int_0^\pi \left\{ \log \left[ \frac{\sqrt{r^2 + b^2 - 2br \cos \phi' + (z - L/2)^2 + b - r \cos \phi'}}{\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z - L/2)^2 + a - r \cos \phi'}} \right] \cos \phi' d\phi' \right. \\
- \log \left[ \frac{\sqrt{r^2 + b^2 - 2br \cos \phi' + (z + L/2)^2 + b - r \cos \phi'}}{\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z + L/2)^2 + a - r \cos \phi'}} \right] \cos \phi' d\phi' \}
\]

(15)

When the second and fourth terms of the series in Equation (9) are used in Equation (5), we find upon integration that each contributes nothing to \( \text{Im} \left[ A_r(r, z) \right] \). In the case of the second term, the reader should confirm that the result follows because \( \int_0^\pi \cos \phi' d\phi' = 0 \), while for the fourth term the contribution that we must evaluate is

\[
\frac{k^3}{6} \frac{\mu_0 I}{4\pi^2} \int_0^b \left[ \int_0^{r^2 + r'^2 - 2rr' \cos \phi' + (z - L/2)^2} dr' \right] \cos \phi' d\phi'.
\]

The integration over \( \phi' \) shows the value of the preceding expression to be zero. Thus, to within the accuracy of our discussion, the real part of \( E_r(r, z) \) is zero. The contribution of the third term in the series to \( \text{Re} \left( A_r \right) \) has been neglected, as it is many orders of magnitude below the contribution of Equation (15), which arises from the first term. Combining the results of the preceding discussion, Equation (15) and Equation (7), we have
\[ E_r(r, z) = -\frac{j k \eta}{4\pi^2} I \int_0^\pi \left[ T(L, \phi') - T(-L, \phi') \right] \cos \phi' d\phi' \text{ V/m}, \]  

(16a)

where

\[ T(L, \phi') = \log \left( \frac{\sqrt{r^2 + b^2 - 2br \cos \phi' + (z - L/2)^2 + b - r \cos \phi'}}{\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z - L/2)^2 + a - r \cos \phi'}} \right). \]

(16b)

and

\[ T(-L, \phi') = \log \left( \frac{\sqrt{r^2 + b^2 - 2br \cos \phi' + (z + L/2)^2 + b - r \cos \phi'}}{\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z + L/2)^2 + a - r \cos \phi'}} \right). \]

(16c)

Because the \( \mathbf{A} \) vector is independent of the variable \( \phi \) and has components only in the radial and axial directions, the relationship \( \mathbf{B} = \nabla \times \mathbf{A} \) tells us that the magnetic field has a component only in the azimuthal direction, and that it is given by

\[ B_\phi(r, z) = -\frac{\partial A_z}{\partial r} + \frac{\partial A_r}{\partial z}. \]

(17)

If we use Equations (4) and (5) to represent the components of \( \mathbf{A} \) in the preceding formula, as well as the series representations described in Equation (9), we conclude that \( B_\phi(r, z) \) has a series expansion with terms varying as \( k^0, k^1, k^2, k^3 \). We recall that \( k = \omega \sqrt{\mu_0 \varepsilon_0} \). Thus, at zero frequency, where \( I \) is now regarded as a direct or quasi-stationary current, the only term present in \( B_\phi(r, z) \) is the one varying as \( k^0 \). We can find this part of \( B_\phi(r, z) \), which we call \( B_{\phi 0}(r, z) \), by a simple application of Ampere’s circuital law around a circle of radius \( r \). The plane of the circle is perpendicular to the \( z \)-axis in Figures 1 and 2, and the axis passes through the center of the circle. The result is
Note that this field is nonzero only within the transformer core: it is the quasi-stationary magnetic-flux density referred to in the hypothetical classroom discussion. Derivation of the preceding result should be easy for students of electromagnetic theory. It is similar to the standard exercise of finding the magnetic-flux density within a coaxial cable carrying a direct current, as described in [3, Sec. 8.2].

To obtain a nontrivial approximation to the magnetic field outside the volume of space containing the transformer core, we seek to include those portions of the vector potential that vary as $k$, $k^2$, and $k^3$. But, as noted previously, inclusion of the second term in the series of Equation (9) (which varies as $k$) contributes nothing to any component of $\mathbf{A}(r, z)$, and therefore contributes nothing to the magnetic-flux density. We thus turn to the third term, which varies as $k^2$. Denoting $A_{z2}(r, \phi)$ and $A_{r2}(r, \phi)$ as the vector components of the magnetic potential varying as $k^2$, we have

\[ B_{\phi 0} = \frac{\mu_0 I}{2\pi r}, \quad a < r < b, \quad |z| < L/2, \tag{18} \]

\[ B_{\phi 0} = 0, \quad \text{if } |z| > L/2, \quad \text{or } 0 \leq r < a, \quad \text{or } r > b. \]
\[ A_{z_2}(r, z) = \frac{\mu_0}{8\pi^2} Ik^2 \left[ \int_0^{L/2} \int_{-L/2}^{L/2} (-R_i + R_0) \, dz' \right] d\phi' \]

\[ = \frac{\mu_0}{8\pi^2} Ik^2 \left[ \int_0^{L/2} \int_{-L/2}^{L/2} \left( -\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z' - z)^2} \right. \right. \]

\[ + \left. \sqrt{r^2 + b^2 - 2br \cos \phi' + (z' - z)^2} \right] dz' \right] d\phi' \]

and

\[ A_{r_2}(r, z) = \frac{\mu_0}{8\pi^2} Ik^2 \left[ \int_a^b \int_{-L/2}^{L/2} (-R_t + R_b) \, dr' \right] \cos \phi' d\phi' \]

\[ = \frac{\mu_0}{8\pi^2} Ik^2 \left[ \int_a^b \int_{-L/2}^{L/2} \left( -\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z' - L/2)^2} \right. \right. \]

\[ + \left. \sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z' + L/2)^2} \right] dr' \right] \cos \phi' d\phi'. \]

We see from Equation (17) that we need the derivative of Equation (19a) with respect to \( r \) and the derivative of Equation (19b) with respect to \( z \). Differentiating under the integral sign in both instances, we arrive at
\[-\frac{\partial A_{z2}(r,z)}{\partial r}\]
\[= \frac{\mu_0}{8\pi^2} I k^2 \left\{ \int_{-L/2}^{L/2} \int_0^\pi \left[ \frac{r - a \cos \phi'}{\sqrt{r^2 + a^2 - 2ar \cos \phi' + (z' - z)^2}} \right] \left[ \frac{r - b \cos \phi'}{\sqrt{r^2 + b^2 - 2br \cos \phi' + (z' - z)^2}} \right] \, dz' \, d\phi' \right\} \]
\[= \frac{\mu_0}{8\pi^2} I k^2 \left\{ \int_{-L/2}^{L/2} \int_0^\pi \left[ \frac{-(z - L/2)}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z - L/2)^2}} \right] \left[ \frac{z + L/2}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z + L/2)^2}} \right] \, dz' \, d\phi' \right\} \]
\[(20a)\]

\[-\frac{\partial A_{r2}(r,z)}{\partial z}\]
\[= \frac{\mu_0}{8\pi^2} I k^2 \left\{ \int_{0}^{a} \int_{b}^{\pi} \left[ \frac{-(z - L/2)}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z - L/2)^2}} \right] \left[ \frac{z + L/2}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi' + (z + L/2)^2}} \right] \, dz' \, d\phi' \right\} \]
\[(20b)\]

The integration over \(z'\) in Equation (20a) can be performed by using the same formula as was used in evaluating Equation (10). The factors \(r - a \cos \phi'\) and \(r - b \cos \phi'\) are constant as we integrate. Thus, we have

\[-\frac{\partial A_{z2}(r,z)}{\partial r}\]
\[= \frac{\mu_0}{8\pi^2} I k^2 \int_0^\pi \left[ \left( r - a \cos \phi' \right) P(a, \phi') - \left( r - b \cos \phi' \right) P(b, \phi') \right] \, d\phi', \]
\[(21a)\]
where the $P$ functions were defined in Equations (13b) and (13c).

The integration over $r'$ in Equation (20b) is done with the same formula as we used in evaluating Equation (14). The factors $(z \pm L/2)$ are constant as we integrate. We obtain

$$\frac{\partial A_{r2}(r, z)}{\partial z} = \frac{\mu_0}{8\pi^2} I k^2 \int_0^{\pi} \left[ -(z-L/2)T(L, \phi') + (z+L/2)T(-L, \phi') \right] \cos \phi' d\phi',$$

(21b)

where the $T$ functions were defined in Equations (16b) and (16c).

Employing Equation (17) as well as Equations (20b) and (21b), we arrive at the following result for that portion of the magnetic flux density varying as $k^2$:

$$B_{\phi2}(r, z) = \frac{\mu_0}{8\pi^2} I k^2 \int_0^{\pi} \left[ (r - a \cos \phi') P(a, \phi') - (r - b \cos \phi') P(b, \phi') \right] d\phi'$$

$$+ \frac{\mu_0 I k^2}{8\pi^2} \int_0^{\pi} \left[ -(z-L/2)T(L, \phi') + (z+L/2)T(-L, \phi') \right] \cos \phi' d\phi'.$$

The first integral in Equation (22) arises from the axial currents, and the second from the radial currents.
A calculation of $B_{\phi 3}(r, \phi)$, which is the portion of the magnetic-flux density varying as $k^3$, shows it to be identically zero. This is because - as noted in our calculation of $E_z$ - the contribution of the $k^3$ terms (from the series in Equation (9)) to $A_z$ results in a constant that drops out in Equation (17). The contribution of the $k^3$ term to $A_r$ is zero, as we observed in our calculation of $E_r$. Returning to Equation (17) completes the argument. Thus, within the constraints that justify the use of Equation (9), Equation (22) yields the magnetic-flux density outside the core of the transformer, while the sum of the fields produced by Equations (18) and (22) yields the flux density inside the core.

**NUMERICAL RESULTS**

All of the numerical results to be presented here are for the conditions $f = 60$ Hz, $a = 1$ m, $b = 2$ m, and $L = 1$m Thus, the core of our transformer was square, with an area of one square meter. The wavelength of a plane wave at 60 Hz is $\lambda = 4.9977 \times 10^6$ m, so that $kb = 2.5144 \times 10^{-6}$. The total current circulating around the core of the toroid was $I = NI_0 = 1$A The numerical integrations were all performed in MATLAB with the aid of a function called QUADL. Figure 3 illustrates $\text{Im}[E_z(r, 0)]$ along the surface $z = 0$ for $0 \leq r \leq 3$ m. The plot was obtained from a numerical evaluation of the far right-hand side of Equation (13a). As we proceed outward from the outer radius $(r = b)$ of the core, the magnitude of the field weakens. The field magnitude also tends
to weaken as we move from the inner radius \((r = a)\) toward the axis of the core. The sign of \(\text{Im}(E_z)\) has opposite values in the regions \(0 \leq r \leq a\) and \(r \geq b\), giving evidence of how the lines of electric field tend to encircle the magnetic flux lines contained in the core. The real part of \(E_z(r, z)\) is independent of the spatial coordinates (see Equation (13a)) and was therefore not plotted. Its value was \(-3.745 \times 10^{-23} \text{V/m}\). Observe that this was about 17 or 18 orders of magnitude smaller than the typical imaginary part shown in Figure 3.

Figure 4 illustrates \(\text{Im}[E_z(r, z)]\) along the cylindrical surfaces \(r = .5, r = 1.5,\) and \(r = 2.5\) for \(|z| \leq 2\). The fields tend to decay rapidly with distance \(|z|\) and are maximum in magnitude, at
$z = 0$ whether we are inside the core ($r = 1.5$) or outside the core ($r = 0.5$ and $r = 2.5$).

Figure 5 shows the value of $\text{Im}[E_r(r, z)]$ along radial paths lying in the planes $z = 0.05, z = 0.25, z = 1.5,$ and $z = 2$. The first two of these paths take us through the core. We see that the field is stronger within the core than outside, and declines rapidly with distance away from the core. The latter two paths do not pass through the core, but they do display their strongest magnitude directly above the core. Some study of Equation (16) shows that $\text{Im}[E_r(r, z)]$ has odd symmetry in the variable $z$. This symmetry also indicates that $\text{Im}(E_r) = 0$ for $z = 0$, and we indeed see from Figure 5 that at $z = 0.5$, the field is already comparatively weak. Recall that use of the four-term series contained in Equation (9) resulted in our finding $\text{Re}[E_r(r, z)] = 0$. 
We turn now to the magnetic-flux density, $B_\phi$. Within the core, $B_\phi$ may be found from the sum of the right-hand sides of Equations (18) and (22). However, here the contribution from the former equation, $B_{\phi 0}$, was found to be about 13 orders of magnitude greater than that of $B_{\phi 2}$, so there was no point in our adding the two and plotting the result. We have chosen to plot only the field $B_{\phi 2}$, given by Equation (22). This is the part of $B_\phi$ that varies as $k^2$; it is the total magnetic-flux density only when we are outside the core. Figure 6 has plots for the three paths $r = 0.5$ m, $r = 2.5$ m, and $r = 1.5$ m for $|z| \leq 2$. A portion of the latter curve $(r = 1.5, |z| < 0.5)$ is shown with asterisks, as it lies inside the core, and represents only a tiny fraction of the flux density there. The portions, of all curves not marked with asterisks represent the total magnetic-flux density vector: a real phasor. It can be seen that the field, $B_{\phi 2}$, is strongest inside the core, and decays rapidly with distance from the core.
We now seek to determine the voltage, $V$, induced in the gap of the loop illustrated in Figure 1 by means of a direct numerical integration of the line integral $\oint E \cdot dl$. Using the numerical values of $E_z$ and $E_r$ that were obtained along the boundary of a rectangular loop described by $r = .5, r = 3, z = \pm 2$) and, for numerical integration, the MATLAB function called TRAPZ, we obtained a potential of $-j5.22616 \times 10^{-5}$ V.

Using the conventional classroom method, we seek this voltage by assuming that all magnetic flux is confined to the core and described by Equation (18). An integration of the flux density in that equation over the cross section of the core yields a phasor
flux of \( \Phi = \frac{\mu_0 IL}{2\pi} \log\left(\frac{b}{a}\right) \). From Faraday’s induction law, the resulting voltage in the gap is simply

\[
-j\omega \Phi = -j\omega \frac{\mu_0 IL}{2\pi} \log\left(\frac{b}{a}\right) = -j k \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{IL}{2\pi} \log\left(\frac{b}{a}\right).
\]

With the values of our parameters, this became

\[-j5.22620 \times 10^{-5} V.\]

The agreement with our line integration of the electric field was remarkably good, and the small disparity was as likely due to the round-off errors in the various numerical integrations in this calculation as it was to the neglect of the magnetic field outside the core in the application of Faraday’s law to the loop. Numerical line integrations of the electric field have been applied to other loops surrounding the core, resulting in equally good agreement with the traditional Faraday method.

**CONCLUSIONS**

The preceding results enable US, as teachers, to say with some confidence, to our class:

The student’s question is valid. There are both electric and magnetic fields outside the core of the transformer. The magnetic field is in the direction of increasing azimuthal angle \( \phi \), while the electric field has \( r \) and \( z \) components. These fields satisfy Maxwell’s equations. What enables us to successfully apply Faraday’s law, as we have done, is that for a transformer used at powerline frequencies and having dimensions of meters (or less), the magnetic-flux density outside the core is many orders of magnitude less than the field inside. Thus, we have made an approximation, but an extraordinarily good one, in taking the flux threading through our loop to be that provided by a simple quasi-stationary formula applicable to the core of the
transformer. Moreover, had we chosen to find the voltage
induced in our loop by a direct line integration of the electric
field, for all practical purposes we would have obtained a result
identical to that obtained with our approximate application of
Faraday's law.

An additional lesson that might be gleaned is that while a
function (like the magnetic-flux density outside the core) may be
so small as to be of little engineering importance, its spatial
derivatives are perhaps multiplied by large factors in Maxwell's
equations. The electric field outside the core can be found from
\[ \mathbf{E} = \frac{\eta}{jk} \mathbf{curl} \mathbf{B} \]. The coefficient in front of the curl equals about
300,000,00 in the present problem. We can appreciate that this
electric field might be significant.

Much of the preceding work - the derivations of the fields and the
computer programming - can be assigned as an exercise for
students in a course in electromagnetic theory. One possible
difficulty is that integral expressions for phasor retarded
potentials (the Helmholtz integrals) are usually not derived until
several weeks after the discussion of Faraday's law.

Finally, some authors (e.g. [5]) argue that if the core of the
transformer is a material of very high permeability, then the
fields external to the core must be zero. However, the kinds of
fields derived here cannot be eliminated with a highly permeable
core. If such an argument were made, then the paradox
described by the hypothetical student would remain unresolved.
Acknowledgment

An anonymous reviewer suggested that I rewrite this paper at a style and level such that it could be readily understood by undergraduates I have tried to follow his or her advice, and I believe that the article is now more useful, although most students will probably still require some guidance. I have profited from discussing this paper with Dr. Robert A. Shore of Hanscom AFB, who has a knowledge of, and interest in, the paradoxes that can arise in electromagnetic theory.